

Transport

- previously developed theory of fluctuations near equilibrium
- = now, \rightarrow transport / relaxation

- prob. is one of Coulomb scattering in stable plasma (i.e. resistivity)

\Rightarrow weak deflection

$\Delta p_{\perp} \sim \int_{-\infty}^{+\infty} dt \frac{e^2}{r^2} \frac{1}{k}$

- Approach via:

- general theory of PDF evolution via series of small, random kicks.
 - \Rightarrow Diffusion, Central Limit Thm.
- Fokker-Planck Eqn. (General)
- Lenard-Balescu Eqn. (extends test particle model)
- FPE + Rosenbluth Potentials

\rightarrow Basics:

Can discretize process (N jumps) with probability of jump by $\frac{1}{N}$ in steps with

1 step back.
↓ (no memory)

$$P_{N+1}(\underline{x}) = \int \rho(\underline{r}) P_N(\underline{x}-\underline{r}) d^d \underline{r}$$

↑
step size

\underline{x} = position,
cumulative

For small steps, expand to 2nd order:

\underline{r} = jump

$$P_{N+1}(\underline{x}) = \int d^d \underline{r} \left\{ P_N(\underline{x}) - \underline{r} \cdot \nabla P_N(\underline{x}) + \frac{1}{2} \underline{r} \cdot \nabla \nabla P_N(\underline{x}) \cdot \underline{r} \right\} \rho(\underline{r}) d^d \underline{r}$$

$$\approx P_N(\underline{x}) + \frac{\langle \underline{r} \cdot \underline{r} \rangle}{2d} \nabla^2 P_N(\underline{x})$$

no bias

→ normalizable ρ , $\langle r^2 \rangle$ exists.

→ no bias

→ thus, have:

$$\frac{P_{N+1}(\underline{x}) - P_N(\underline{x})}{\Delta t} = \frac{\langle r^2 \rangle}{2d \Delta t} \nabla^2 P_N$$

diffusion equation!

$$\left\{ \begin{array}{l} \frac{\partial P}{\partial t} = D \nabla^2 P \\ D = \langle r^2 \rangle / 2d \Delta t \end{array} \right.$$

$\rho \rightarrow \rho$

$\rho(x, 0) = \delta(x)$

SD $\frac{\partial \rho}{\partial t} = -Dk^2 \rho$

FT

$\rho(k, t) = e^{-Dk^2 t} \rho(k, 0)$
 $= e^{-Dk^2 t} \cdot 1$

SD Inverse \Rightarrow

$\rho(x, t) = \frac{e^{-x^2/4Dt}}{(4\pi Dt)^{1/2}}$

\rightarrow pdf for position, in time

C.C. $\rho(k, t) = e^{-Dk^2 t}$
 $= e^{-D(k_1^2 + k_2^2 + k_3^2) t}$

SD $\rho(x, t) = \int e^{i(k_1 x_1 + k_2 x_2 + k_3 x_3 + \dots)} e^{-D(k_1^2 + k_2^2 + k_3^2) t} dk_1 dk_2 \dots dk_d$

\therefore = above.

$\rho(x, t) \sim \frac{1}{t^{d/2}} e^{-x^2/4Dt}$

$x^2 = \underline{x} \cdot \underline{x}$

then, re-discretizing:

$$P_N(x) = \frac{\exp[-dx^2 / 2\langle v^2 \rangle N]}{(2\pi \langle v^2 \rangle N/d)^{d/2}}$$

$$P_N(x) \sim \frac{e^{-d x^2 / 2\langle v^2 \rangle N}}{(\langle v^2 \rangle N)^{d/2}}$$

$P_N(x)$:

- converges, time asymptotically, to Gaussian, with width $\sim N^{1/2} \langle v^2 \rangle^{1/2} \sim N^{1/2} v_{rms}$
- $\sim N^{-d/2}$.

\Rightarrow quick demo of Central Limit Theorem.

Soln. Ti :

Let X_1, X_2, \dots be a sequence of:

- independent
- identically distributed

random variables each with mean μ and variance σ^2

i.e. $\bar{X}_i = \mu$

$$\langle (X_i - \bar{X}_i)^2 \rangle = \sigma^2$$

When the distribution of:

$$\frac{X_1 + X_2 + \dots + X_n - n\mu}{\sigma \sqrt{n}}$$

$n \rightarrow \infty$
 \rightarrow normal
 (Gaussian)

i.e.

$$P \left\{ \frac{X_1 + X_2 + \dots + X_n - n\mu}{\sigma \sqrt{n}} \in \epsilon \right\}$$

$$= \int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$$

Note: Holds for all sequences

- independent \rightarrow no correlations
- identically distributed
 i.e. no 'special' steps, \rightarrow intermittently.
- σ^2 exists \rightarrow no fat tails

Related: Law of Large #s

Let X_1, X_2, \dots be a sequence of random variables having a common distribution and let

$$E(X_i) = \mu$$

↑
expectation

Then, with probability 1:

$$\frac{X_1 + X_2 + \dots + X_n}{n} \rightarrow \mu$$

as $n \rightarrow \infty$

In simple terms;

$n \rightarrow \infty$, X_i : rdm variable

- LLN: Average conv. to $E(X_i)$

- CLT: ^{Sum} Distribution \rightarrow Gaussian
with \sqrt{n} , centroid $n\mu$.